

# Notes on normed algebras, 3

Stephen William Semmes  
Rice University  
Houston, Texas

Let  $A$  be a countably-infinite commutative semigroup with an identity element  $0$ . Thus  $A$  is equipped with a binary operation  $+$  which is commutative and associative, and  $x + 0 = x$  for all  $x \in A$ .

If  $f_1, f_2$  are two complex-valued functions on  $A$ , then we can try to define the convolution of  $f_1, f_2$  by

$$(1) \quad (f_1 * f_2)(z) = \sum_{x+y=z} f_1(x) f_2(y),$$

where more precisely the sum is taken over all  $x, y \in A$  such that  $x + y = z$ . This sum makes sense if at least one of  $f_1, f_2$  has finite support, which is to say that it is equal to 0 for all but at most finitely many  $x \in A$ . In particular, if we restrict our attention to functions with finite support, then we get a nice commutative algebra.

For each  $a \in A$  define  $\delta_a$  to be the function on  $A$  which is equal to 1 at  $a$  and to 0 at all other points in  $A$ . For any function  $f$  on  $A$ , the convolution of  $f$  with  $\delta_0$  is equal to  $f$  again. The convolution of  $\delta_a, \delta_b$  is equal to  $\delta_{a+b}$  for all  $a, b \in A$ . Thus the semigroup  $A$  is embedded into the convolution algebra of functions on  $A$  with finite support in such a way that the semigroup operation on  $A$  corresponds exactly to convolutions of functions. Functions on  $A$  with finite support are the same as the finite linear combinations of the  $\delta_a$ 's, so that convolution for any two functions on  $A$  with finite support is determined by convolution of the  $\delta_a$ 's and linearity.

Consider the homomorphisms of this convolution algebra onto the complex numbers. In other words, these are the linear mappings  $\phi$  from the vector space of functions on  $A$  with finite support into the complex numbers such that  $\phi(f_1 * f_2)$  is equal to  $\phi(f_1)$  times  $\phi(f_2)$  for all functions  $f_1, f_2$ .

$f_2$  on  $A$  with finite support and  $\phi(\delta_0) = 1$ . If  $\phi$  is such a homomorphism, then  $\Phi(a) = \phi(a)$  defines a homomorphism from  $A$  into the multiplicative semigroup of complex numbers such that  $\Phi(0) = 1$ . Conversely, if  $\Phi$  is a homomorphism of  $A$  into the multiplicative semigroup of complex numbers, which means that  $\Phi(a+b) = \Phi(a)\Phi(b)$  for all  $a, b \in A$ , and if  $\Phi(0) = 1$ , then we get a homomorphism  $\phi$  from the convolution algebra of functions on  $A$  with finite support onto the complex numbers by reversing the process.

For instance, suppose that  $A$  is the semigroup of nonnegative integers. Then for each complex number  $z$  we get a homomorphism  $\Phi$  from  $A$  into the multiplicative semigroup of complex numbers such that  $\Phi(0) = 1$ , by setting  $\Phi(j) = z^j$  when  $j \geq 1$ . If  $A$  is the group of integers under addition, then for each nonzero complex number  $z$  we get a homomorphism  $\Phi$  from  $A$  into the multiplicative semigroup of complex numbers by setting  $\Phi(j) = z^j$  for all integers  $j$ .

Suppose again that  $A$  is a countably infinite commutative semigroup with identity element 0. A function  $f$  on  $A$  is said to be summable if  $\sum_{a \in A} |f(a)|$  is finite, and in this event we write  $\|f\|_1$  for the sum. If  $f_1, f_2$  are two functions on  $A$  and at least one of  $f_1, f_2$  is summable and the other is bounded, then the convolution  $f_1 * f_2$  can be defined as before, and is a bounded function on  $A$ . If  $f_1, f_2$  are two summable functions on  $A$ , then the convolution  $f_1 * f_2$  is also summable, and  $\|f_1 * f_2\|_1$  less than or equal to the product of  $\|f_1\|_1$  and  $\|f_2\|_1$ . Thus the vector space of summable functions on  $A$  becomes a commutative algebra with respect to convolution.

Let  $\Phi$  be a homomorphism from  $A$  into the multiplicative semigroup of complex numbers such that  $\Phi(0) = 1$ . Assume also that  $\Phi$  is bounded, which implies that  $|\Phi(a)| \leq 1$  for all  $a \in A$ . Then we can define a linear mapping  $\phi$  from the vector space of summable functions on  $A$  into the complex numbers by saying that  $\phi(f)$  is equal to  $\sum_{a \in A} \Phi(a) f(a)$  for all summable functions  $f$  on  $A$ . Thus  $\phi(\delta_0) = 1$ , and one can check that  $\phi(f_1 * f_2)$  is equal to the product of  $\phi(f_1)$  and  $\phi(f_2)$  when  $f_1, f_2$  are summable functions on  $A$ . We also have that  $|\phi(f)| \leq \|f\|_1$  for all summable functions  $f$  on  $A$ .

For instance, if  $A$  is the semigroup of nonnegative integers, and  $z$  is a complex number such that  $|z| \leq 1$ , then we get such a homomorphism  $\Phi$  from  $A$  into the multiplicative semigroup of complex numbers by setting  $\Phi(0) = 1, \Phi(j) = z^j$  when  $j \geq 1$ . If  $A$  is the commutative group of all integers, then for each complex number  $z$  with  $|z| = 1$  we obtain such a homomorphism by setting  $\Phi(j) = z^j$  for all  $j$ .

Let  $A$  be a countably infinite commutative semigroup with identity ele-

ment 0 again. Assume that for each  $a \in A$  there are only finitely many pairs  $a_1, a_2 \in A$  such that  $a_1 + a_2 = a$ . In this case convolution can be defined as before for any two functions  $f_1, f_2$  on  $A$ , and the space of complex-valued functions on  $A$  becomes an algebra with respect to convolution. For instance, this condition holds when  $A$  is the semigroup of nonnegative integers. However, we cannot convert a homomorphism  $\Phi$  from  $A$  into the multiplicative semigroup of complex numbers into a homomorphism  $\phi$  from the convolution algebra into the complex numbers except in trivial situations where  $\Phi(a) = 0$  for all but finitely many  $a \in A$ .

One can consider continuous versions of these notions as well. For instance, let  $n$  be a positive integer, and let  $A$  be a closed convex cone in  $\mathbf{R}^n$ . Thus  $A$  is a closed subset of  $\mathbf{R}^n$  which contains 0 and which has the property that  $x + y \in A$  when  $x, y \in A$  and  $tx \in A$  when  $x \in A$  and  $t$  is a nonnegative real number. Let us also assume that  $A$  is not contained in a lower dimensional subspace of  $\mathbf{R}^n$ . This is equivalent to saying that the linear span of  $A$  is all of  $\mathbf{R}^n$ , and to  $A$  containing a nonempty open set.

In general, for two functions  $f_1, f_2$ , the convolution  $(f_1 * f_2)(x)$  is defined by integrating  $f_1(y)$  times  $f_2(x - y)$  with respect to  $y$ . For this one needs suitable conditions on  $f_1, f_2$ , e.g., both are continuous and one has compact support, or one is integrable and the other is bounded. If both  $f_1, f_2$  have support in  $A$ , then the convolution does too.

If  $\zeta$  is an element of  $\mathbf{C}^n$ , then we get a continuous homomorphism from  $\mathbf{R}^n$  as a commutative group with respect to addition into the nonzero complex numbers as a multiplicative group by setting  $\Phi(x)$  equal to  $\exp(\zeta \cdot x)$ , where  $\zeta \cdot x$  is defined to be  $\sum_{j=1}^n \zeta_j x_j$ . This leads to a linear mapping from continuous functions with compact support on  $\mathbf{R}^n$  into complex numbers by putting  $\phi(f)$  equal to the integral of  $\Phi(x)$  times  $f(x)$ . If  $f_1, f_2$  are two continuous functions with compact support on  $\mathbf{R}^n$ , then the convolution  $f_1 * f_2$  is also a continuous function with compact support, and we get an algebra with respect to convolution. One can check that  $\phi(f_1 * f_2) = \phi(f_1)\phi(f_2)$  in this case, so that  $\phi$  defines a homomorphism from the convolution algebra of continuous functions on  $\mathbf{R}^n$  into the complex numbers.

If  $f_1, f_2$  are integrable on  $\mathbf{R}^n$ , then the convolution  $f_1 * f_2$  is defined as an integrable function on  $\mathbf{R}^n$ , and we get a convolution algebra again. To get a homomorphism into the complex numbers we can start with a homomorphism  $\Phi$  from  $\mathbf{R}^n$  as an additive group into the multiplicative group of nonzero complex numbers by setting  $\Phi(x)$  equal to  $\exp(i\eta \cdot x)$ , where now  $\eta$  is an element of  $\mathbf{R}^n$  so that  $\Phi$  is bounded, and in fact  $|\Phi(x)| = 1$  for all  $x \in \mathbf{R}^n$ .

If  $f$  is integrable on  $\mathbf{R}^n$ , then we define  $\phi(f)$  to be the integral of  $\Phi(x)$  times  $f(x)$ . As before,  $\phi(f_1 * f_2)$  is equal to the product of  $\phi(f_1)$  and  $\phi(f_2)$  when  $f_1, f_2$  are integrable, so that  $\phi$  defines a homomorphism from the convolution algebra into the complex numbers.

Now let us restrict our attention to integrable functions  $f_1, f_2$  which are equal to 0 on  $\mathbf{R}^n \setminus A$ . Suppose that  $\zeta = \xi + i\eta \in \mathbf{C}^n$ , where  $\xi, \eta \in \mathbf{R}^n$ . Suppose also that  $-\xi \in A^*$ , which is to say that  $\xi \cdot x \geq 0$  for all  $x \in A$ . Then  $\Phi(x) = \exp(\zeta \cdot x)$  defines a bounded homomorphism from  $A$  as an additive semigroup into the multiplicative group of nonzero complex numbers. One can put  $\phi(f)$  equal to the integral of  $\Phi(x)$  times  $f(x)$  where  $f$  is an integrable function supported on  $A$  to get a homomorphism from the convolution algebra of integrable functions supported on  $A$  into the complex numbers.

## References

- [1] Y. Katznelson, *An Introduction to Harmonic Analysis*, second edition, Dover, 1976.
- [2] W. Rudin, *Fourier Analysis on Groups*, Wiley Classics Library, 1990.
- [3] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series **32**, Princeton University Press, 1971.